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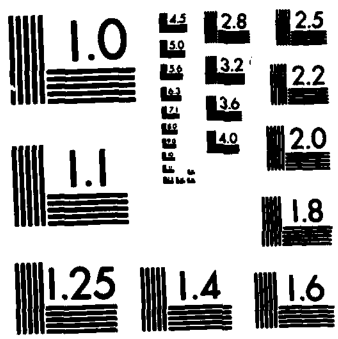

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APPLICATION OF A FINITE DIFFERENCE METHOD TO THE TRANSONIC AIRFOIL PROBLEM

THESIS

AFIT/GAE/AA/79D-4

Michael Patrick Burke
Captain USAF

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This study presents the development and evaluation of a numerical solution technique used to analyze the two dimensional transonic airfoil problem. The full potential equation of motion and the irrotationality condition are used for the governing equations. A coordinate transformation is applied to the governing equations to limit the domain of the problem and for easier application of the boundary conditions. The method of lines is then used to reduce the equations from partial to ordinary differential equations. Solutions obtained using this

numerical solution technique are compared to published data for both incompressible and compressible flowfield cases. It is concluded that the solution technique developed in this study is accurate and efficient when analyzing subcritical flowfields around circular airfoil shapes.

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AN APPLICATION OF A FINITE DIFFERENCE METHOD TO THE TRANSONIC AIRFOIL PROBLEM

THESIS

Presented to the Faculty of the School of Engineering

of the Air Force Institute of Technology

Air University

in Partial Fulfillment of the

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Master of Science

by

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Captain USAF

Graduate Aeronautical Engineering

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PREFACE

I would like to thank all of the people whose assistance made this study possible. In particular, I wish to thank my advisor, Major Steven Koob, for his efforts in educating, assisting and supervising me in my struggle to complete this work. I also wish to express my gratitude to Captain Hugh Briggs and Professor Urmila Ghia who took the time to review my work and offer their most helpful comments and encouragement.

Michael P. Burke

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List of Symbols

c	Local speed of sound
c_∞	Freestream speed of sound
M	Local Mach number
M_∞	Freestream Mach number
$\bar{Q}$	Local velocity
Q_r	Radial component of local velocity
Q_θ	Tangential component of local velocity
q	Nondimensional local velocity
(r, θ)	Cylindrical coordinates
u	Nondimensional radial velocity
v	Nondimensional tangetial velocity
γ	Specific heat ratio
Φ	Potential function
ϕ	Nondimensional potential function
η	Transformation coordinate for r

Abstract

This study presents the development and evaluation of a numerical solution technique used to analyze the two dimensional transonic airfoil problem. The full potential equation of motion and the irrotationality condition are used for the governing equations. A coordinate transformation is applied to the governing equations to limit the domain of the problem and for easier application of the boundary conditions. The method of lines is then used to reduce the equations from partial to ordinary differential equations. Solutions obtained using this numerical solution technique are compared to published data for both incompressible and compressible flowfield cases. It is concluded that the solution technique developed in this study is accurate and efficient when analyzing subcritical flowfields around circular airfoil shapes.

APPLICATION OF A FINITE DIFFERENCE METHOD TO THE TRANSONIC AIRFOIL PROBLEM

I. Introduction

Background

The development of an accurate and efficient numerical solution technique for the 2-D transonic airfoil problem is one of the greater challenges of modern aerodynamic research. The problem is physically and mathematically complex hence the difficulty in developing a model which accurately reflects what is physically occurring in the transonic flowfield. Nevertheless, recent trends in aircraft design and performance requirements have created the need for accurate solutions to the transonic airfoil problem.

To appreciate the difficulty in developing a solution technique for this problem one must have an understanding of the nature of the physical phenomena involved. In the transonic airfoil problem there is a mixture of subsonic ($M<1$) and supersonic ($M>1$) flow regions giving rise to what is termed supercritical flow (subcritical flow occurs when the flow is subsonic at all points in the flowfield; critical flow occurs when $M=1$ at one point in the flowfield). Subsonic flow accelerating around the airfoil surface creates regions where $M>1$. Expansion waves leaving the airfoil surface will reflect off the sonic line as compression waves (figure 1). Some of the compression waves will reach the airfoil surface while others will reach and strengthen the shock wave. The shock wave is strongest at the airfoil

surface and weakens to zero strength near the top of the supersonic flow region. Further complicating the flowfield is the interaction of the shock wave and the boundary layer. If the shock wave is strong enough the boundary layer may become separated from the airfoil surface thus extending the viscous forces beyond the thin boundary layer and significantly altering the inviscid flowfield. Separation of the boundary layer at the airfoil trailing edge may also alter the flowfield. In addition to creating a local deviation from inviscid flow theory, trailing edge separation can cause a change in circulation about the airfoil and thus a change in lift. This in turn can affect the shock wave location. Finally, trailing edge separation may interact with a shock wave and change the pressure distribution about the airfoil surface.

It should be expected that a phenomenon as complicated as the transonic airfoil problem requires a mathematical model equally complex if the problem is to be modelled accurately. The governing equations for the transonic airfoil problem do indeed contain several outstanding mathematical complexities when the flowfield is supercritical. First, if the problem is posed as a steady-state problem the governing differential equations are of the mixed elliptic-hyperbolic type, being elliptic for subsonic flow regions and hyperbolic for supersonic flow regions. Second, the equation of motion for transonic flow is nonlinear. As the freestream Mach number approaches unity the nonlinear terms become large.

Thus near Mach one the equations cannot be linearized if the problem is to be accurately modelled. A third problem exists when attempting to account for the presence of shock waves. Shock waves exist in flowfields as physical discontinuities and must be modelled as such. Including shock waves in a model requires the application of the Rankine-Hugoniot relationships across the shock wave as well as satisfying the original governing equations throughout the rest of the flowfield.

Two fundamental approaches have commonly been used to analyze the transonic airfoil problem. One is called the inverse approach. Here the shock wave data is specified and the airfoil shape and details of the flow within the supersonic region are unknown. This approach has the attractive features of using little computer time and requiring relatively little computer storage space. The shock characteristics are iterated in this method until the solution for a specified airfoil shape is determined. The other approach is called the direct method. Here the airfoil shape is given and all of the details of the flow within the supersonic region are unknown. The direct method requires much more computing time and storage space than the inverse method.

It stands to reason that a set of governing equations which accurately models the transonic airfoil problem will be very complex. However, mathematical models which omit or approximate some of the phenomena in the flowfield will be less complex and easier to work with. For example, for flows

with extensive separated regions inviscid flow theory must be combined with viscous theory to correctly predict the flowfield. This is a complex model which can be difficult to analyze. With attached flows or flowfields containing only locally separated regions, inviscid theory alone may give useful approximations, thus allowing use of a simpler model. Finally, assuming shockless flow allows a simpler model to be used by requiring fewer equations to describe the flowfield. Such approximations may be made in the early development of a solution technique to simplify the task of proving the validity of the technique. Once it has been shown to yield useful approximations, model refinements can be made to have the model more accurately represent the flowfield.

Exact solutions to the transonic airfoil problem are available for only the simplest of cases (ref 4). However, several numerical methods have been developed which yield approximate solutions.

Numerical Methods Currently Available

Three types of numerical methods have generally been used to analyze the transonic airfoil problem. One of these is the time-dependent method. Here are some of the difficulty in solving a mixed elliptic-hyperbolic differential equation is overcome by retaining the time dependent terms in the governing equation. That is, the problem is posed as an unsteady one. This makes the equations hyperbolic with respect to time. The original boundary value problem is

recast as an initial-value/boundary value problem. A finite difference scheme is then used to integrate the equations forward in time. If the boundary conditions are time-independent or become so, the solution will coverge to a steady-state solution equivalent to that for the original steady boundary value problem. Using the time-dependent approach assures that the problem is properly posed mathematically. Magnus and Yoshihara (ref 16) have been particularly successful with this method, having applied it to several airfoil shapes. Their results were the first reported solutions for flows with imbedded shock waves which compared favorably with experimental data. Grossman and Moretti (ref 7) have also used time-dependent methods with notable success.

A second type of numerical solution technique is the relaxation methods. Relaxation methods are a standard procedure for solving boundary value problems described by elliptic partial differential equations. Relaxation can be viewed as the recasting of a elliptic differential equation into a parabolic or hyperbolic differential equation. Murman and Cole (ref 20), Murman and Krupp (ref 21), Jameson and South (ref 12), and Garabedian and Korn (ref 6) have, among others, successfully applied relaxation methods to the transonic airfoil problem. The essential feature of their work is the accounting for the local physical nature of the flowfield while applying a finite differencing formula to the governing equations describing the flowfield. Most

relaxation methods have been applied using transonic small disturbance theory. Steger and Lomax (ref 22) have applied relaxation using both small disturbance theory and full inviscid theory.

Finally, various approximation methods have been applied to the transonic airfoil problem. These methods employ mathematical simplifications to reduce partial differential equations to ordinary differential equations. The approach developed in this study is of this type. Approximations methods have been successfully applied using both the integral method and the method of integral relations.

The integral method recasts the governing equations into a set of integral relations through the use of Green's theorem. An approximation is then used to represent the variation of a dependent variable in one direction. The equation is then integrated in that direction leaving an ordinary differential equation with respect to the other independent variable. This method has not been found to consistently produce results comparing favorably with experimental data.

The method of integral relations differs from the above method in that the flowfield is divided into strips, thus discretizing one of the independent variables. Separate approximations for the dependence of the dependent variables with respect to the discretized variable are then used for each strip. A matching condition is imposed at the strip boundaries. This method has been successfully used by Tai (ref 23), Melnik and Ives (ref 18), Holt and Masson (ref 9),

Crown (ref 2) and Dorodnitsyn (ref 3). Informative summaries of numerical work on the transonic airfoil problem have been written by Murman (ref 19), Taylor (ref 24), and Yoshihara (ref 25).

Statement of Problem

The objective of this study is to develop and evaluate a simple numerical solution technique applicable in both the subcritical and supercritical regimes. The method should provide a good approximation to the flow field being modelled and should use relatively little computer time. The method will first be applied to the simple case of incompressible flow around the unit circle airfoil. The method will then be tested on compressible flow around the unit circle airfoil. Through the application of an appropriate transformation the circular airfoil results will be applied to Joukowski airfoil shapes for the incompressible flowfield case. To reduce the complexity of the problem only shockless flowfields will be studied. Flowfields will be assumed to be inviscid and all flows will be irrotational and isentropic.

Approach to the Problem

The full inviscid potential theory and the irrotationality condition will provide the set of governing equations to be used in this study. The boundary conditions to be applied to the problem are no flow normal to the airfoil at the airfoil surface and uniform flow at infinity. A major facet of the problem formulation will be a coordinate transformation

applied to the governing equations and boundary conditions. The new coordinate system will simplify the geometry of the problem and will make application of the infinity boundary condition easier. Of the three types of numerical solution techniques mentioned it was decided that one of the approximation methods would be used in this study. This choice was made based upon the desire to keep the numerical model relatively simple. It has been shown by McCracken (ref 17) that useful results can be obtained when approximate methods are used in a simplified system model. Specifically the method of lines, a version of the method of integral relations, will be used in this study. This solution technique has been used by, among others, McCracken (ref 17), Klunker, Davis and South (ref 15), Hamilton (ref 8), and Melnik and Ives (ref 18). The method of lines will be used to reduce the set of governing partial differential equations to a set of ordinary first order differential equations. This new set will then be numerically integrated to yield a velocity distribution along the airfoil surface. The solution technique will be evaluated by comparing the computed velocity distributions with published data.

The unit circle airfoil was chosen because the exact solution for incompressible flow around it is known. It thus presented itself as one way in which the accuracy of the solution technique developed in the study could be tested. Another reason the unit circle airfoil was chosen is that the Joukowski transformation may be applied to it thus allowing airfoil-like shapes to be studied.

II. Development of the Solution Technique

There are several methods available with which to analyze the 2-D transonic airfoil problem. Regardless of the method used certain choices must be made which will define the approach used to work the problem. First a set of governing equations must be chosen. Boundary conditions must also be specified. Second, a coordinate system must be chosen within which to work the problem. Third, a numerical solution technique must be developed to solve the set of governing equations.

When choosing a set of governing equations a trade-off is made between the accuracy desired in the system model and the level of complexity acceptable. For inviscid flow analysis one of three sets of relationships have commonly been used to mathematically model the 2-D transonic airfoil problem. They differ primarily in how each accounts for the presence of shock waves.

One set consists of the isentropic relationship

$$\frac{P}{P_o} = \left(\frac{\rho}{\rho_o}\right)^{\gamma} \tag{1}$$

along with the unsteady form of the conservation equations for mass and momentum

$$\begin{aligned} \frac{\partial \rho}{\partial t} + (\nabla \cdot \rho \overline{Q}) &= 0 \\ \frac{\partial \overline{Q}}{\partial t} + (\overline{Q} \cdot \nabla)\,\overline{Q} &= -\nabla\left(\frac{P}{\rho}\right) \end{aligned} \tag{2}$$

These are applied throughout the flowfield except across the shock, where the Rankine-Hugoniot relationships are used.

The number of equations in this formulation however makes it difficult to work with. Such a set of equations has been used by Grossman and Moretti (ref 7).

A second set of equations consists of the steady-state form of the previous set written in divergence form. This set can be applied across a shock wave thus avoiding the need for special relationships across the shock wave. However, this set of equations is of the mixed type. This set has been employed in 2-D transonic airfoil analysis by Tai (ref 23).

A third set of equations may be fomulated starting with the assumption that a good approximation to the flowfield can be obtained without including shock waves in the model. This assumption is compatible with the requirement for the flow to be isentropic since isentropic flow implies either weak shocks or no shocks present in the flowfield. Based upon the desire to start with as simple a model as possible it was decided to use this set and neglect shock waves. Therefore the equations used in this study model the 2-D transonic airfoil problem assuming steady, inviscid, irrotational, isentropic flow. Given these assumptions, inviscid flow theory and the irrotationality condition were used to define a set of governing equations in cylindrical coordinates. The nondimensionalized equations are stated here. The interested reader is referred to the Appendix for the details of their development. The irrotationality equation is

$$\frac{\partial u}{\partial \theta} = v + r \frac{\partial v}{\partial r} \qquad (4)$$

and the inviscid full potential equation of motion is

$$\frac{\partial v}{\partial \theta} = \frac{2\ Euv}{B-Au^2-Dv^2}\ \frac{r\partial v}{\partial r} - \frac{B-Du^2-Av^2}{B-Au^2-Dv^2}\ \frac{r\partial u}{\partial r} - u \tag{5}$$

For flow around the unit circle airfoil the boundary conditions were that the normal velocity component at the airfoil surface equal zero and that uniform flow conditions exist at infinity. In nondimensionalized form the zero velocity boundary condition was written

$$u(1,\theta) = 0 \tag{6}$$

and the infinity boundary condition was written

$$u^2(\infty,\theta) + v^2(\infty,\theta) = 1 \tag{7}$$

The decision to use the unit circle airfoil made the choice of a cylindrical coordinate system obvious. It was noted though, that this choice could be improved by applying the simple inversion coordinate transformation

$$\eta = \frac{1}{r} \tag{8}$$

The effect of this transformation is shown in figure 2. The advantages gained by its application are two-fold. First, the domain of the problem is reduced to a finite region. Second, the transformation solves the problem of where to apply the infinity boundary condition. The application of this coordinate transformation to the governing equations and boundary conditions is shown in the Appendix. The results are that the irrotationality condition becomes

$$\frac{\partial u}{\partial \theta} = v-\eta\ \frac{\partial v}{\partial \eta} \tag{9}$$

and the equation of motion becomes

$$\frac{\partial v}{\partial \theta} = \frac{Av^2+Du^2-B}{Au^2+Dv^2-B}\,\eta\frac{\partial u}{\partial \eta} + \frac{2Euv}{Au^2+Dv^2-B}\,\eta\frac{\partial v}{\partial \eta} - u \tag{10}$$

The boundary condition of no normal velocity component at the airfoil surface remains unchanged by the transformation. The infinity boundary condition becomes

$$u^2(0,\theta) + v^2(0,\theta) = 1 \tag{11}$$

There remained the choice of which numerical solution technique to use to solve this set of equations. It was decided to use the method of lines because of its ease of application to the governing equations used in this study. The method of lines is used to reduce the partial differential equations to first order ordinary differential equations. This is done by allowing the derivatives with respect to η to be represented by a finite difference approximation. As illustrated in Figure 3 the η domain is divided by N concentric lines equally spaced between $\eta = 0$ and $\eta = 1$ such that

$$\eta_i = \frac{i-1}{N+1},\ i = 2, 3, \ldots, N, N+1 \tag{12a}$$

$$\text{with } \eta_1 = 0 \text{ and } \eta_{N+2} = 1 \tag{12b}$$

Along the N lines the derivatives with respect to η are approximated by a central difference formula. Details of the formula and how it was used in the study are shown in the study are shown in the Appendix. The result is that along the i^{th} line the governing equations are written

$$\frac{du_i}{d\theta} = v_i - \frac{i-1}{N+1}\left(\frac{v_{i+1}-v_{i-1}}{\frac{2}{N+1}}\right) \tag{13}$$

$$\frac{dv_i}{d\theta} = \frac{Av_i^2+Du_i^2-B}{Au_i^2+Dv_i^2-B}\left(\frac{i-1}{N+1}\right)\left(\frac{u_{i+1}-u_{i-1}}{\frac{2}{N+1}}\right) +$$

$$\frac{2Eu_iv_i}{Au_i^2+Dv_i^2-B}\left(\frac{i-1}{N+1}\right)\left(\frac{v_{i+1}-v_{i-1}}{\frac{2}{N+1}}\right) - u_i \tag{14}$$

Writing these equations for each of the N lines results in a set of 2N simultaneous equations involving 2N+4 unknowns, u_i and v_i, i=1, 2, ..., N, N+1, N+2. Clearly four more relationships are needed to make the problem determinate. Three of these are obtained from the boundary conditions as

$$u(0,\theta) = -\cos\theta \tag{15}$$

$$v(0,\theta) = \sin\theta \tag{16}$$

$$u(1,\theta) = 0 \tag{17}$$

The fourth relationship is derived by applying the irrotationality condition along the airfoil surface. Here it is known that the radial velocity component u is equal to zero for all θ. This implies that along the airfoil surface $\frac{\partial u}{\partial \theta}$ is equal to zero for all θ. Thus, for $\eta=1$ (i.e., along the airfoil surface) one can write

$$\frac{\partial u}{\partial \theta} = 0 = v - \frac{\eta \partial v}{\partial \eta} \tag{18}$$

After substituting for η and applying a backward difference formula to approximate the derivative with respect to η, equation (18) is rewritten

$$v_{N+2} = \left(\frac{4N+4}{3N+1}\right) v_{N+1} - \left(\frac{N+1}{3N+1}\right) v_N \tag{19}$$

To summarize, the problem now consists of 2N first order nonlinear ordinary differential equations and one algebraic relationship along with three boundary conditions, two of these applied at $\eta = 0$ for all θ, and the third applied at $\eta = 1$ for all θ. All that remained was to develop a scheme with which to integrate these equations and to state the initial conditions to be used.

For the sake of simplicity it was decided to use a "canned" integration package called DVERK which was available in the International Mathematical and Scientific Library (IMSL) of computer programs. DVERK finds approximations to the solution of a system of first order ordinary differential equations with initial conditions. The program uses Runge-Kutta formulas of order 5 and 6. As applied in this study, DVERK, given a set of initial conditions, would integrate the 2N ordinary differential equations with respect to θ by some amount $\Delta\theta$. The integration stepsize could be chosen by the user to be whatever size was necessary to obtain satisfactory results. In this study $\Delta\theta$ was typically one degree. The product of each integration step consisted of the normal and tangential velocity components on each line at the angle the problem has been integrated to. The algebraic relationship developed by applying the irrotationality condition at the airfoil surface was then used to compute the velocity at the airfoil surface at that same angle. The velocities on each

line as well as at the airfoil surface then served as the initial conditions for DVERK to integrate another amount $\Delta\theta$. By integrating with repsect to θ from zero to ninety degrees a velocity profile along the airfoil surface would be generated.

Next, a scheme for providing a set of correct initial conditions at $\theta = 0°$ had to be developed. This was not necessary for the incompressible flowfield case since exact solutions for this case are available. However, a scheme had to be developed which would provide a correct set of initial conditions at $\theta = 0°$ for compressible flowfield cases. The following discussion outlines the scheme developed to obtain these initial conditions for compressible flowfield cases.

Correct initial conditions were obtained by starting with an estimated set of initial conditions at $\theta = 0°$, a freestream Mach number slightly greater than zero (typically 0.1), and using from three to seven lines. Using this set of estimated initial conditions the governing equations were integrated to $\theta = 90°$. After marching ninety degrees, normal and tangential velocity components were available on each line at $\theta = 90°$ as well as at the airfoil surface. For the flowfields analyzed in this study the normal velocity components are zero at $\theta = 90°$. This ideal velocity profile was never realized however after having numerically integrated one time to $\theta = 90°$ when starting with an estimated set of initial conditions at $\theta = 0°$. It was assumed however that there did exist some velocity distribution at $\theta = 0°$ which

if used as a set of initial conditions, would allow the integration to yield a correct velocity distribution (i.e. one matching the ideal) at $\theta = 90°$. The problem was to find this velocity profile.

It was decided that this could best be accomplished by systematically adjusting the normal velocities at $\theta = 0°$ to obtain normal velocities of zero at $\theta = 90°$. A first order Newton-Raphson iterative scheme was used to determine the changes required. Using this scheme, each normal velocity component at $\theta = 0°$ was, in turn, varied a small amount from its original estimated value. DVERK was then used to integrate the governing equations through ninety degrees. The change in each normal velocity component at $\theta = 90°$ as a result of varying each normal velocity component at $\theta = 0°$ was thus calculated. These changes were used to compute rates of change of velocities at $\theta = 90°$ as functions of small velocity changes at $\theta = 0°$. The rates were then used in an iterative Newton-Raphson scheme to determine the amount each normal velocity component at $\theta = 0°$ had to be varied from its original estimated value to produce normal velocity components of value zero at $\theta = 90°$.

If the problem of computing a correct normal velocity profile at $\theta = 0°$ were linear, only one iteration of the Newton-Raphson scheme would be needed. However, the problem is non-linear and several iterations of the Newton-Raphson scheme were needed. Use of the Newton-Raphson scheme imposed one constraint upon the original estimate of the normal

velocity profile at $\theta = 0°$. Since it is a linear method being used to solve a non-linear problem, the original estimate could not be too far from the final values. This constraint was overcome by simply insuring that only small changes were needed to the original estimate. This was accomplished by starting with the known exact velocity profile for incompressible flow and solving a flowfield problem at only a slightly higher Mach number. Then using this new profile the Mach number was increased slightly again and the problem solved again. This incremental process was repeated until a solution was obtained for the desired freestream Mach number.

In this study, a flowfield was analyzed starting with the incompressible velocity profile at $\theta = 0°$ and solving for the velocity profile (and correct initial conditions) at $M = 0.1$. Results for $M = 0.1$ were then used as a starting point in analyzing the flowfield at $M = 0.2$, and so forth up to the desired Mach number. This scheme proved to be convergent for subcritical, critical and slightly supercritical flowfields. For these cases the normal velocities at $\theta = 90°$ could be made to approach zero using as few as four lines.

III. Results

The results and conclusions of the study are presented in order of increasing complexity of the flowfield being analyzed. The results for the case of the unit circle airfoil in an incompressible flowfield are first presented followed by the results for a symmetrical Joukowski airfoil in an incompressible flowfield. Results for the unit circle airfoil in a compressible flowfield are then presented.

Incompressible Flowfield Results

Incompressible flow about the unit circle airfoil was chosen as the first application of the solution technique for several reasons. First, this simple case has a known exact solution (ref 14). The results from the solution technique developed in this study could be compared to this exact solution. This comparison would be useful as an indication of the accuracy of the solution technique developed. A second reason lay in the fact that the governing equations for this case were known to have a high degree of numerical stability due to their linearity in the variables u and v. The irrotationality condition is always linear in these variables. However, the equation of motion is linear only for the case of incompressible flow. For this case the equations are

$$\frac{\partial u}{\partial \theta} = v - \eta \frac{\partial v}{\partial \eta} \qquad (20)$$

$$\frac{\partial v}{\partial \theta} = \eta \frac{\partial u}{\partial \eta} - u \qquad (21)$$

Starting with the boundary conditions as previously stated the solution technique was numerically intergrated around the airfoil from $\theta = 0$ to $\theta = 90°$. Because of the symmetry of the problem all of the flowfield was known if the flowfield in this quandrant was known. The numerical integration generated a velocity distribution along the airfoil surface. This data was then compared to the exact solution. Figure 4 shows the velocity distribution at the surface of the unit circle airfoil in an incompressible flowfield along with the exact solution. Three lines were used for this case. It was found that for incompressible flow as few as three lines were needed for accurate results. Increasing the number of lines did not increase the accuracy when analyzing incompressible flowfields.

Having solved the case of incompressible flow about the unit circle airfoil, the Joukowski transformation was applied to that flowfield to analyze the case of incompressible flow about a Joukowski airfoil. Figures 5 and 6 show the velocity distribution for two typical Joukowski airfoil profiles as obtained by the solution technique developed in this study.

Compressible Flowfield Results

Unit circle airfoil surface velocity profiles were obtained over a range of Mach numbers. Figure 7 shows the results for a subcritical flowfield case. Figure 8 shows the

results for the critical flowfield case where M = 0.42.

Compressible flowfields with freestream Mach numbers up to 0.45 were successfully studied. Attempts to analyze flowfields with freestream Mach numbers higher than 0.45 failed however. Neither the Newton-Raphson linear scheme nor a nonlinear least squares scheme (ref 5) would produce convergent results. Varying the numerical integration step size and the number of lines did not solve the problem. According to previously published reports in which the method of lines was used (ref 13), the difficulty is due to the inherent instability present in the governing equations when the method of lines is used with a central difference approximation for the derivatives. Central differencing is appropriate for approximating derivatives in elliptic equations (subcritical flow) where this differencing can yield convergent results. But for hyperbolic differential equations (supercritical flow) central difference approximations are unstable and will not yield convergent results.

Figure 9 shows the velocity distribution for the highest freestream Mach number that the solution technique would successfully analyze. At this Mach number, $M_\infty = 0.45$, it should be noted that the maximum local Mach number is in the flowfield is 1.45. Therefore the solution technique can be said to be useful in analyzing supercritical flowfields to at least a limited extent.

IV Summary and Recommendations

Summary

After comparing results obtained using the solution technique developed in this study with published data it was concluded that the solution technique produces accurate results when analyzing subcritical flow around circular airfoil shapes. The method also gives good results for slightly supercritical flowfields. However, when analyzing supercritical flowfields around circular shaped airfoils the method breaks down for $M_\infty > .45$. It is concluded that the finite differencing scheme used in this study, i.e. the method of lines with central differencing, is appropriate when the system governing equations are of the elliptic type. This occurs for subcritical flows around circular airfoil shapes. The scheme is however incorrect when analyzing supercritical flowfields because central differencing alone cannot account for the local nature of the flowfield in regions of supercritical flow.

Recommendations

The following recommendations are proposed for any future work in modelling flowfields with the method of lines:

1. A finite differencing and integration scheme using the method of lines should be developed which would account for the local physical nature of flowfields in the transonic

regime. The integration scheme should be capable of sensing when the flowfield is transitioning from subsonic to supersonic flow. The scheme should then be capable of applying the appropriate differencing formulas in response to the type of flowfield sensed.

2. An inversion transformation such as the type used in this study should be developed which will allow the solution technique used to be applied to a variety of airfoil shapes. The benefits of such a transformation include easier application of the flowfield boundary conditions as well as extending the utility of the technique.

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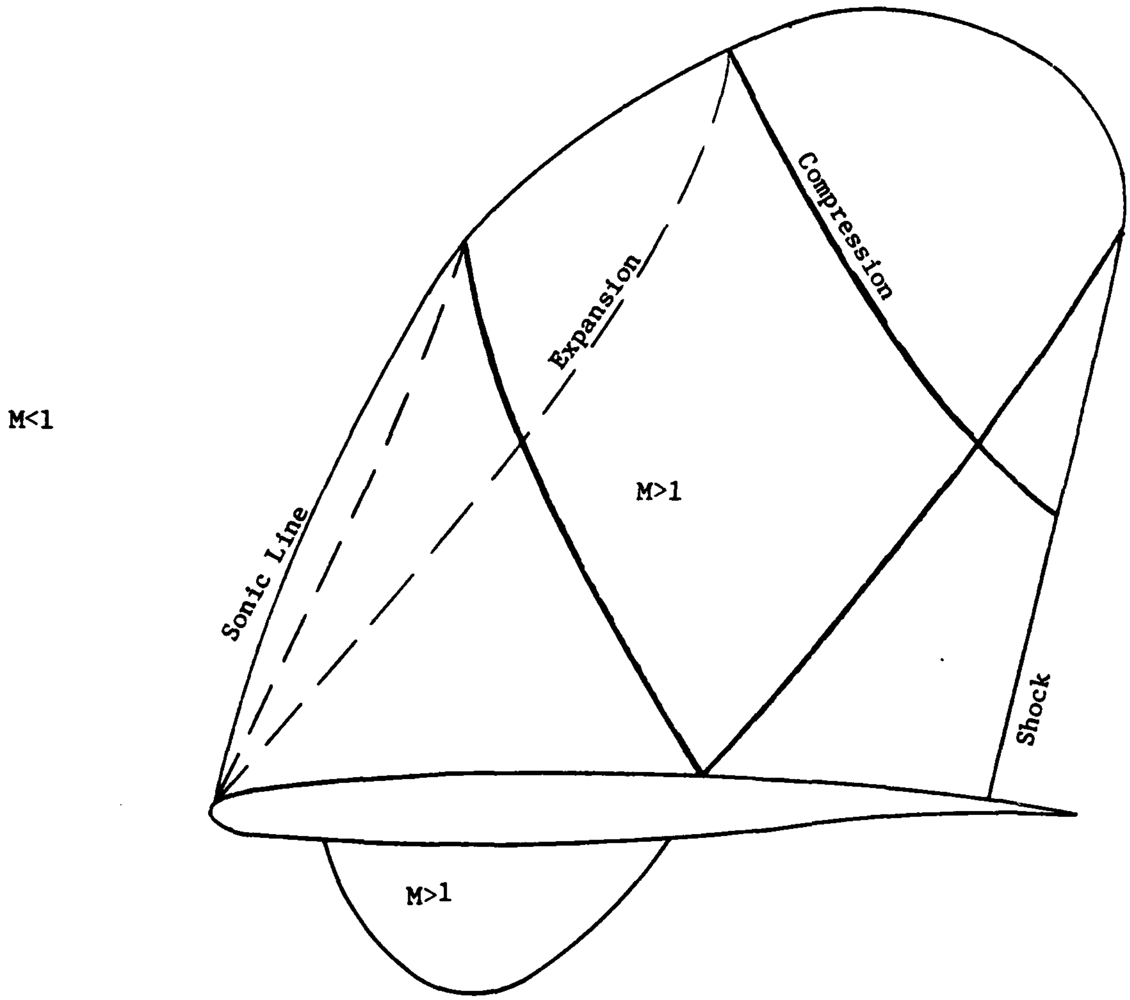

Figure 1

Schematic Diagram of Supercritical Flow Around an Airfoil

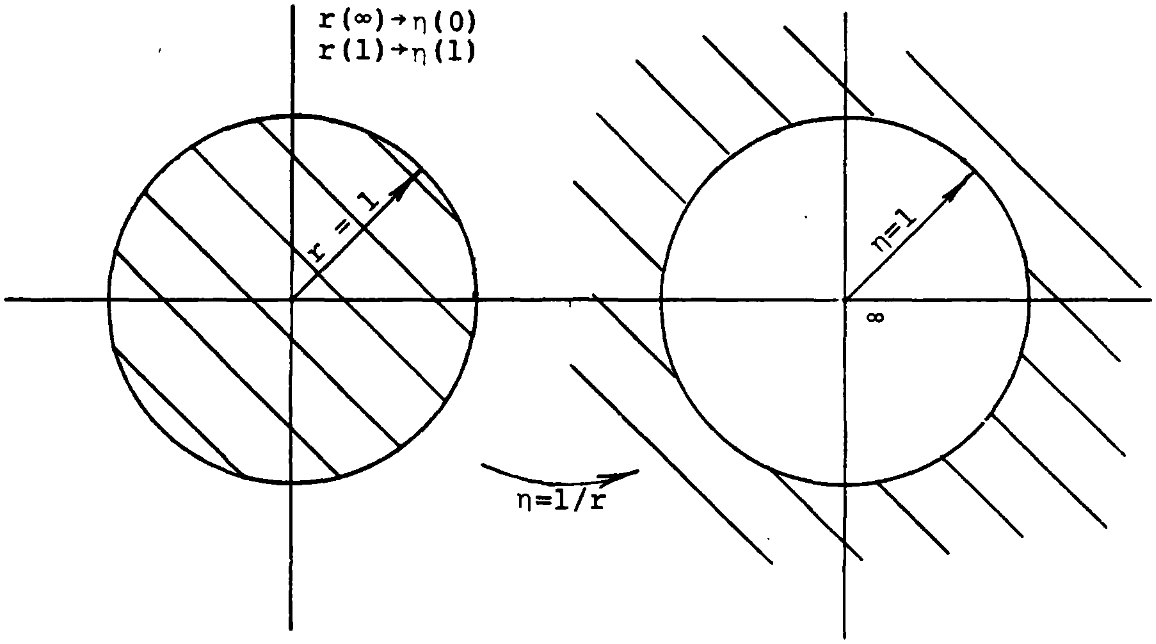

Figure 2

Effect of the Inversion Transformation on the Unit Circle and Surrounding Space

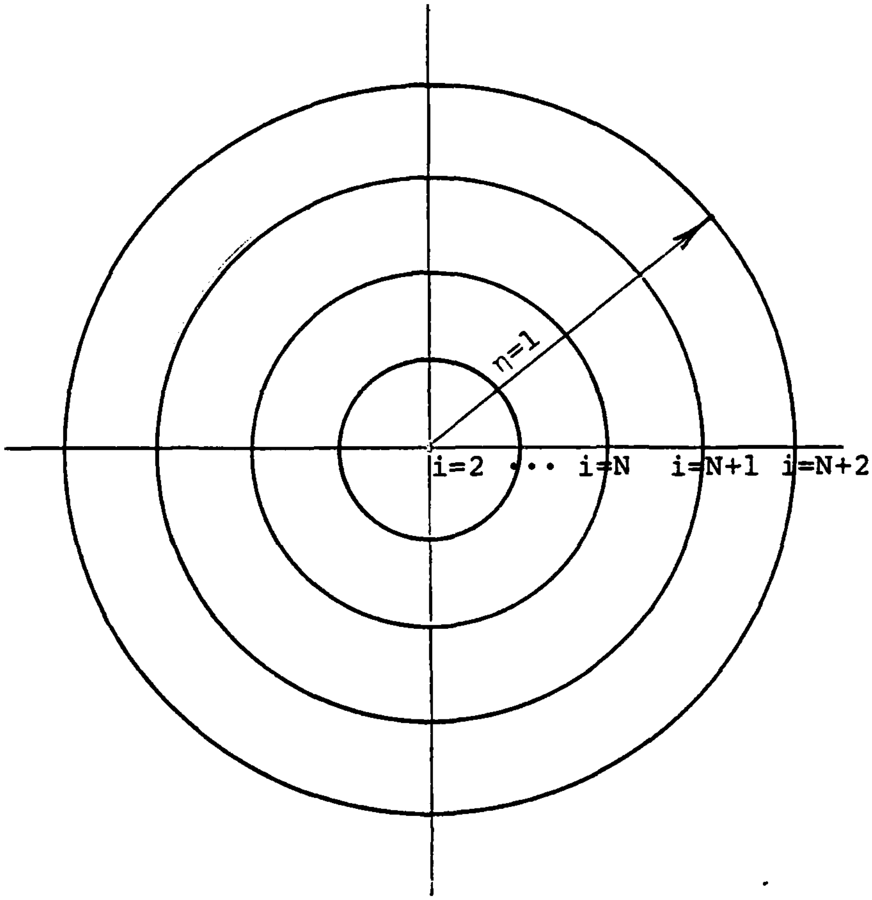

Figure 3

Discretization of the η-Domain

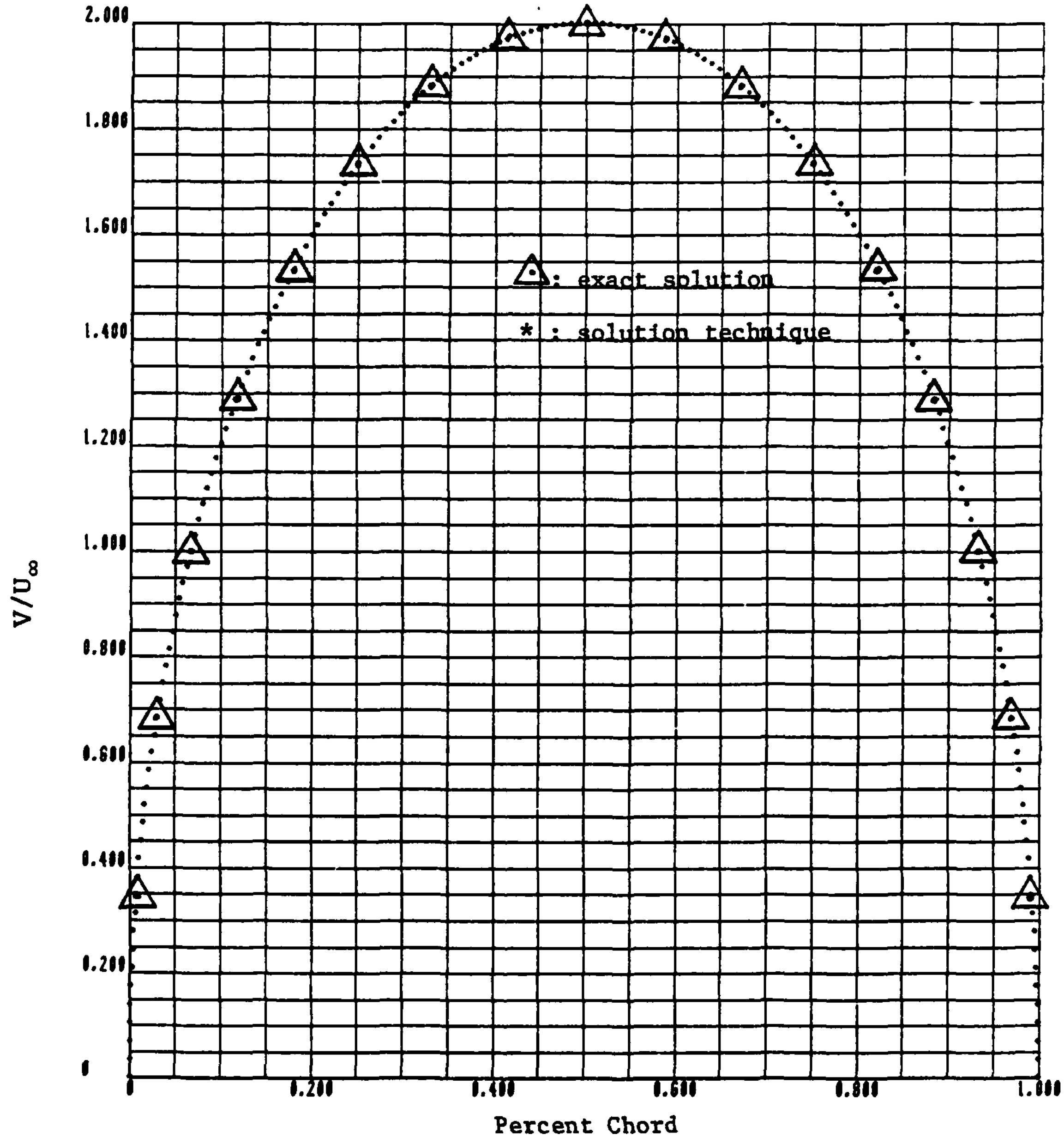

Figure 4

Solution Obtained for the Velocity Distribution at the Surface of the Unit Circle for $M_\infty=0$ as Compared to the Exact Solution. N=3

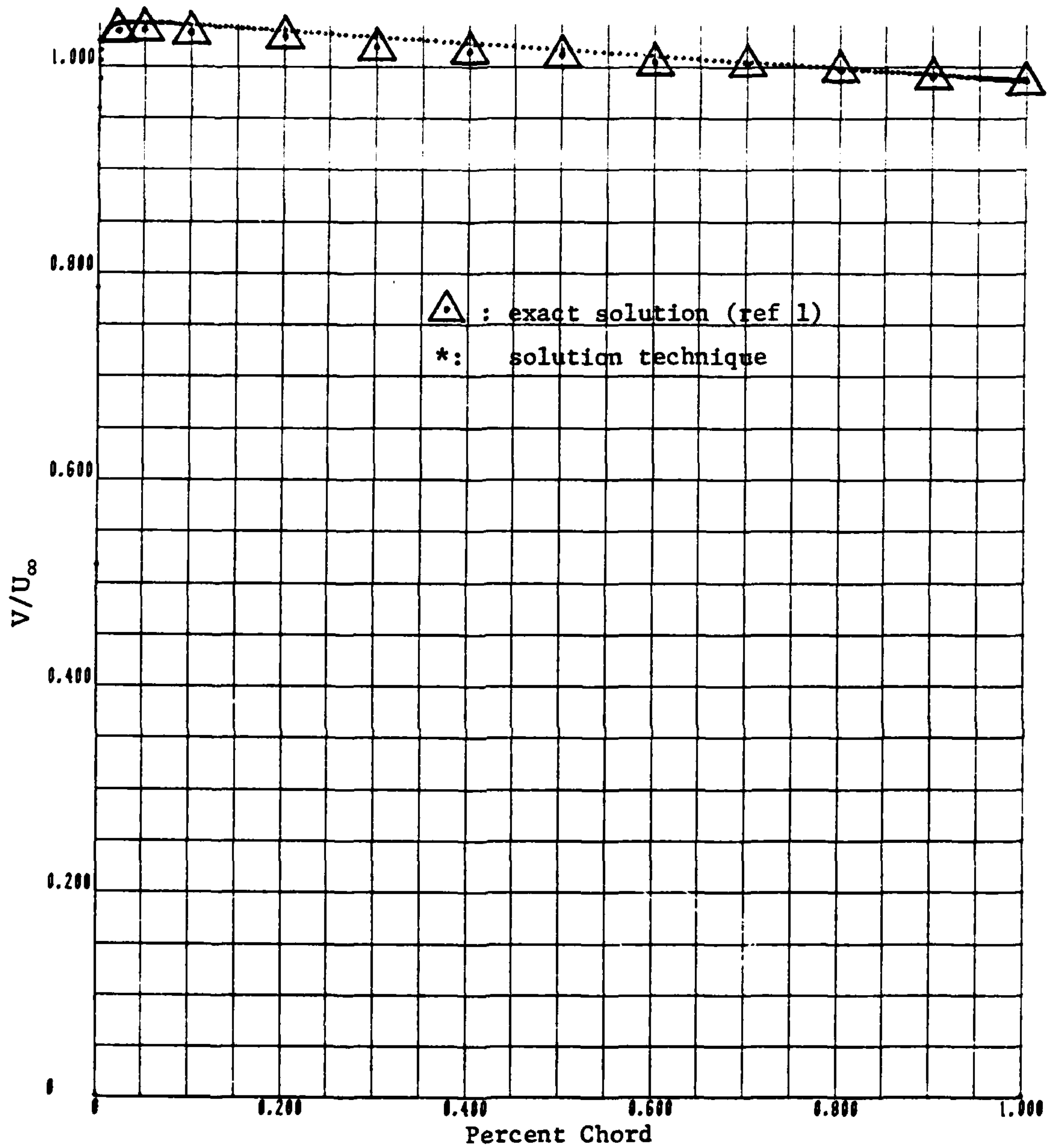

Figure 5

Solution Obtained for the Velocity Distribution at the Surface of a 2% Thick Symmetrical Joukowski Airfoil for M_∞=0 as Compared to the Exact Solution. N=3

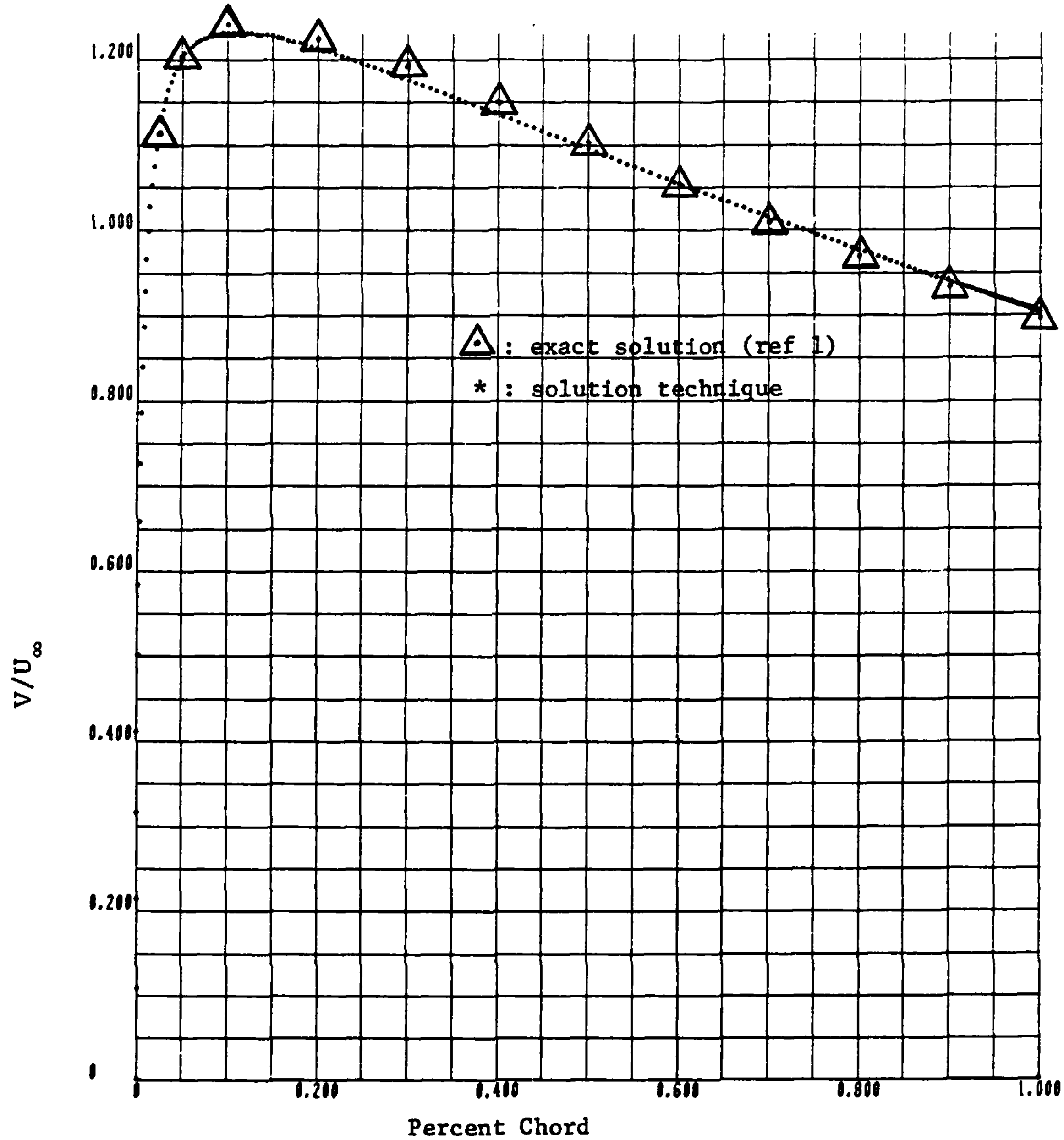

Figure 6

Solution Obtained for the Velocity Distribution at the Surface of a 14% Thick Symmetrical Joukowski Airfoil for M_∞=0 as Compared to the Exact Solution. N=3

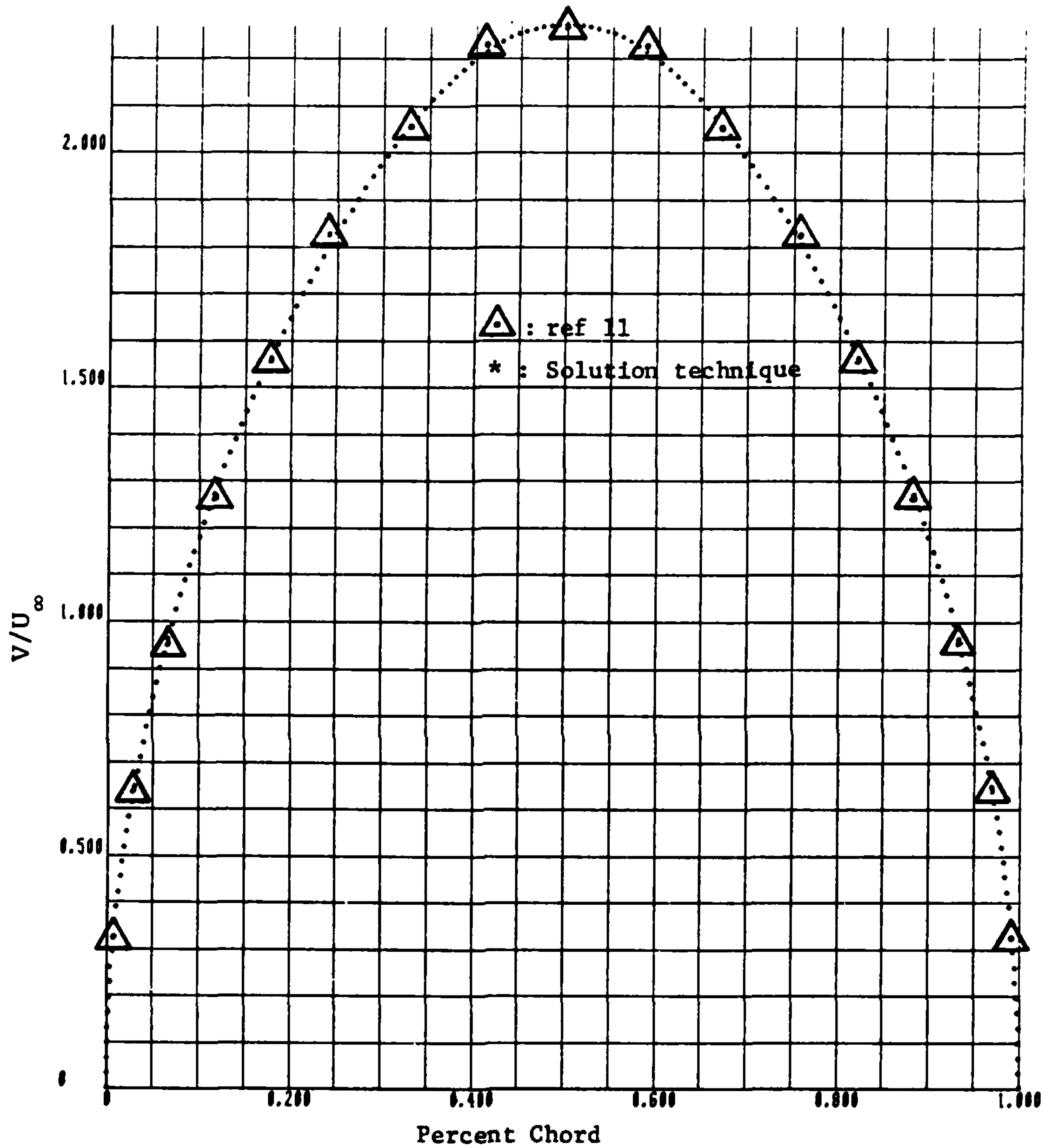

Figure 7

Solution Obtained for the Velocity Distribution at the Surface of the Unit Circle for the Sub-Critical Flowfield Case of M_∞=0.4 as Compared to Published Date. N=4

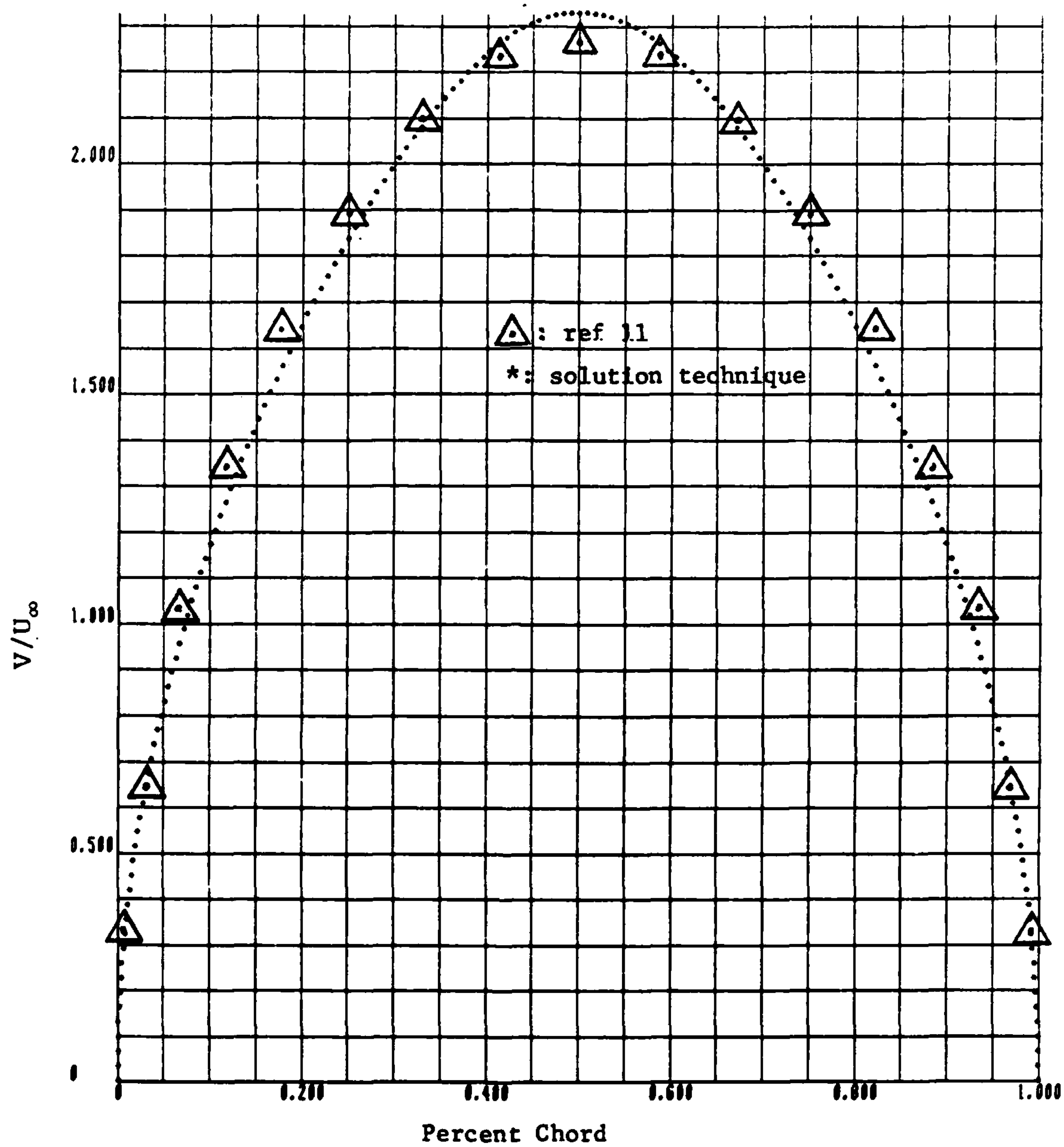

Figure 8

Solution Obtained for the Velocity Distribution at the Surface of the Unit Circle for the Critical Flowfield Case of M_∞=0.42 as Compared to Published Data. N=4

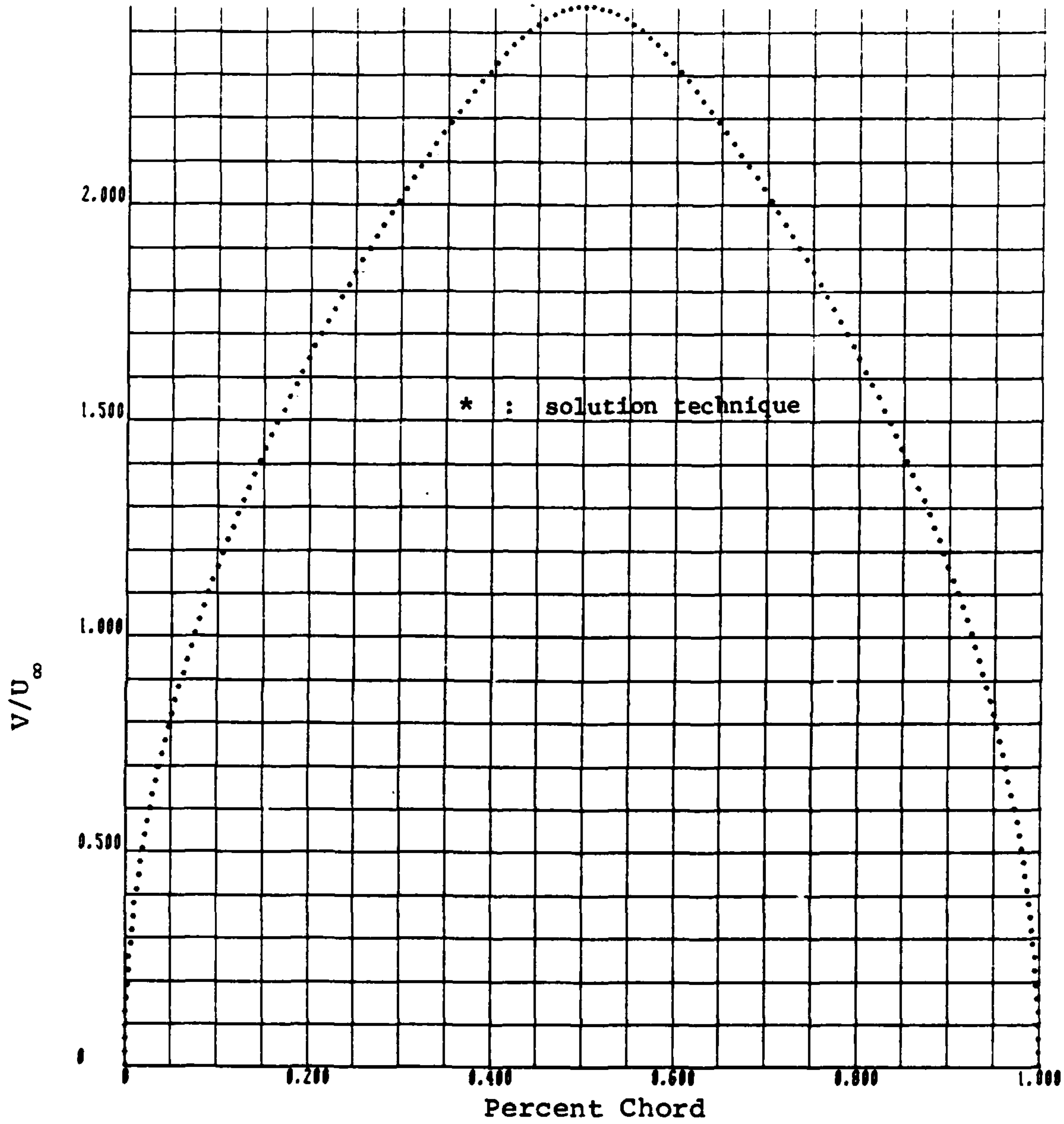

Figure 9

Solution Obtained for the Velocity Distribution at the Surface of the Unit Circle Airfoil for the Supercritical Flowfield Case of M_∞=0.45. N=4

Appendix A

Details of the Solution Technique Development

This appendix provides a detailed account of the development of the solution technique used in this study. The governing equations and boundary conditions are first developed. Then the equations are nondimensionalized and arranged into a more concise form. Next a coordinate transformation is applied to the nondimensionalized governing equations and boundary conditions. Finally, the method lines is used to reduce the governing equations from partial to ordinary differential equations.

Development of the Governing Equations

The set of governing equations used in this study are used to model steady, inviscid, irrotational, isentropic flow. As a starting point the continuity equation

$$\nabla \cdot (\rho\bar{Q}) = 0 \qquad (22)$$

the equation of motion

$$(\bar{Q} \cdot \nabla)\,\bar{Q} = -\nabla\left(\frac{p}{\rho}\right) \qquad (23)$$

and the equation of state

$$p = k\rho^{\gamma} \qquad (24)$$

are used from which the final equations will be derived. The speed of sound, c, given by $c^2 = \left.\frac{\partial p}{\partial \rho}\right|_s$, may be introduced

into equation (23) which, when then combined with equation (22), gives

$$\bar{Q} \cdot \nabla \left(\frac{Q^2}{2}\right) - c^2 \nabla \cdot \bar{Q} = 0 \tag{25}$$

letting $Q^2 = \bar{Q} \cdot \bar{Q}$.

The definition of c^2 may again be used to combine equations (23) and (24).

The resulting equation can be integrated once to give

$$\frac{c^2}{\gamma-1} + \frac{Q^2}{2} = \frac{c_\infty^2}{\gamma-1} + \frac{Q_\infty^2}{2} \tag{26}$$

where the constant of integration is chosen to match the conditions of undisturbed flow, i.e., $\bar{Q} = (-Q_\infty \cos\theta,$ $Q_\infty \sin\theta)$ and $c = c_\infty$.

With the assumption of irrotationality, a potential function ϕ may be introduced into equations (25) and (26) with

$$\frac{\partial\phi}{\partial r} = Q_r = -Q_\infty \cos\theta \quad \text{and} \tag{27a}$$

$$\frac{1}{r}\frac{\partial\phi}{\partial\theta} = Q_\theta = Q_\infty \sin\theta \tag{27b}$$

Equation (25) may then be rewritten as

$$\nabla^2\phi = \frac{1}{2c^2}\left(\frac{\partial\phi}{\partial r}\frac{\partial Q^2}{\partial r} + \frac{1}{r^2}\frac{\partial\phi}{\partial\theta}\frac{\partial Q^2}{\partial\theta}\right) \tag{28}$$

with $$Q^2 = \left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\phi}{\partial\theta}\right)^2 \tag{29}$$

Equation (26) may also be rewritten as

$$c^2 = c_\infty^2 - \frac{(\gamma-1)}{2}(Q^2 - Q_\infty^2) \tag{30}$$

Equation (28) is the equation of motion used for compressible potential flow. The assumption of irrotational flow allows the use of a potential function. The irrotationality condition is expressed as

$$\nabla \times \bar{Q} = 0 \tag{31}$$

One boundary condition to be used is the assumption of uniform flow at infinity. This boundary condition will be applied in the form

$$Q_\infty^2 = (-Q_\infty \cos\theta)^2 + (Q_\infty \sin\theta)^2$$
$$= Q_r^2 + Q_\theta^2 = U^2 + V^2 \qquad (32)$$

The second boundary condition is that there be no normal velocity component at the airfoil surface, i.e.

$$U^2 = 0 \qquad (33)$$

along the airfoil surface.

Nondimensionalization

Defining $\phi = \frac{\Phi}{Q_\infty}$, $q = \frac{Q}{Q_\infty}$ and $M_\infty = \frac{Q_\infty}{c_\infty}$ (34)

the equation of motion may be written in nondimensional form as

$$\left(1 - M_\infty^2 \frac{(\gamma-1)}{2} (q^2-1)\right) \nabla^2\phi = \frac{M_\infty^2}{2} \left(\frac{\partial\phi}{\partial r}\frac{\partial q^2}{\partial r} + \frac{1}{r^2}\frac{\partial\phi}{\partial\theta}\frac{\partial q^2}{\partial\theta}\right) \qquad (35)$$

with $$q^2 = \frac{\partial\phi^2}{\partial r} + \frac{1}{r^2}\left(\frac{\partial\phi}{\partial\theta}\right)^2 \qquad (36)$$

and $$\frac{c^2}{Q_\infty^2} = \frac{1}{M_\infty^2}\left(1 - M_\infty^2\left(\frac{(\gamma-1)}{2} (q^2-1)\right)\right) \qquad (37)$$

Equation (37) when expanded can be written

$$\left(\left(1-M_\infty^2\right)\left(\frac{(\gamma-1)}{2} (q^2-1)\right)\right)\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}\right) =$$
$$M_\infty^2\left[\left(\frac{\partial\phi}{\partial r}\right)^2\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r^2}\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial\theta}\frac{\partial^2\phi}{\partial r\partial\theta} - \frac{1}{r^3}\frac{\partial\phi}{\partial r}\left(\frac{\partial\phi}{\partial\theta}\right)^2 + \right.$$
$$\left.\frac{1}{r^4}\left(\frac{\partial\phi}{\partial\theta}\right)^2\frac{\partial^2\phi}{\partial\theta^2}\right] \qquad (38)$$

Assuming potential flow allows one more simplification to be made by allowing

$$\frac{\partial \phi}{\partial r} = u \quad \text{and} \quad \frac{1}{r}\frac{\partial \phi}{\partial \theta} = v \tag{39}$$

with u being the nondimensional radial velocity component and v being the nondimensional tangential velocity component. Applying these definitions to equation (38) yields, after some rearrangement,

$$\left[M_\infty^2 \frac{(\gamma-1)}{2} + 1\right] \frac{\partial u}{\partial r} + \left[M_\infty^2 \frac{(\gamma-1)}{2} + 1\right] \frac{1}{r}\frac{\partial v}{\partial \theta} +$$

$$\left[M_\infty^2 \frac{(\gamma-1)}{2} + 1\right] \frac{u}{r} - \left[M_\infty^2 \frac{(\gamma-1)}{2} + M_\infty^2\right] u^2 \frac{\partial u}{\partial r} - \left[M_\infty^2 \frac{(\gamma-1)}{2}\right]\frac{u^2}{r} \frac{\partial v}{\partial \theta} -$$

$$\left[M_\infty^2 \frac{(\gamma-1)}{2}\right] \frac{u^3}{r} - \left[M_\infty^2 \frac{(\gamma-1)}{2}\right] v^2\frac{\partial u}{\partial r} - \left[M_\infty^2 \frac{(\gamma-1)}{2} + M_\infty^2\right] \frac{v^2}{r} \frac{\partial v}{\partial \theta} -$$

$$\left[M_\infty^2 \frac{(\gamma-1)}{2} - M_\infty^2\right] \frac{uv^2}{r} - 2M_\infty^2 \frac{uv}{r} \frac{\partial u}{\partial \theta} = 0 \tag{40}$$

Finally, this equation may be rearranged into the form

$$\frac{\partial v}{\partial \theta} = \frac{Av^2+Du^2-B}{B-Dv^2-Au^2}\, r\, \frac{\partial u}{\partial r} + \frac{2Euv}{B-Dv^2-Au^2} \frac{\partial u}{\partial \theta} - u \tag{41}$$

with $A = M_\infty^2 \frac{(\gamma-1)}{2}$ (42a)

$B = M_\infty^2 \frac{(\gamma-1)}{2} + 1$ (42b)

$D = M_\infty^2 \frac{(\gamma-1)}{2} + M_\infty^2$ and (42c)

$E = M_\infty^2$ (42d)

Equation (41) is the equation of motion in the form used in this study. The irrotationality condition may be written in similar form as

$$\frac{\partial u}{\partial \theta} = v + r\frac{\partial v}{\partial r} \tag{43}$$

Nondimensionalizing the infinity boundary condition, equation (32), yields $\frac{Q_\infty^2}{Q_\infty^2} = \frac{(-Q_\infty \cos\theta)^2}{Q_\infty^2} + \frac{(Q_\infty \sin\theta)^2}{Q_\infty^2}$

$$= \frac{U^2}{Q_\infty^2} + \frac{V^2}{Q_\infty^2} = u^2 + v^2 = 1 \tag{44}$$

The nondimensional surface boundary condition simply becomes

$$\frac{U}{U_\infty} = u = 0 \tag{45}$$

Application Of The Coordinate Transformation

To simplify the geometry of the problem and the application of the boundary conditions the coordinate transformation $\eta = \frac{1}{r}$ is employed. First, the transformation laws are developed. Assume that some function $f(r,\theta) = F(\eta,\theta)$ exists, whose value doesn't change with a change in coordinate systems. The following equations can then be applied:

$$\frac{\partial f}{\partial r} = \frac{\partial F}{\partial \eta}\frac{\partial \eta}{\partial r} + \frac{\partial F}{\partial \theta}\frac{\partial \theta}{\partial r} \tag{46}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial F}{\partial \eta}\frac{\partial \eta}{\partial \theta} + \frac{\partial F}{\partial \theta}\frac{\partial \theta}{\partial \theta} \tag{47}$$

Note that $\frac{\partial \eta}{\partial r} = \frac{-1}{r^2}$ (48a)

$$\frac{\partial \eta}{\partial \theta} = \frac{\partial \theta}{\partial r} = 0, \text{ and} \tag{48b}$$

$$\frac{\partial \theta}{\partial \theta} = 1 \tag{48c}$$

Substituting equations (48) into equations (46) and (47) yields $\frac{\partial f}{\partial r} = \frac{-1}{r^2}\frac{\partial F}{\partial \eta} = -\eta^2 \frac{\partial F}{\partial \eta}$ (49)

$$\frac{\partial f}{\partial \theta} = \frac{\partial F}{\partial \theta} \qquad (50)$$

Equations (49) and (50) are the transformation laws needed. Applying them to equations (41) and (43) gives

$$\frac{\partial v}{\partial \theta} = \frac{Av^2+Du^2-B}{Au^2+Dv^2-B}\,\frac{\eta\partial u}{\partial \eta} + \frac{2Euv}{Au^2+Dv^2-B}\,\frac{\eta\partial v}{\partial \eta} - u \qquad (51)$$

$$\frac{\partial u}{\partial \theta} = v - \frac{\eta\partial v}{\partial \eta} \qquad (52)$$

Application of the Method of Lines

By using the method of lines the derivatives with respect to η are represented by a finite difference approximation. Hornbeck (ref 10) shows how the finite difference approximations are derived as well as the error terms implicit in their use.

After application of the coordinate transformation the new domain is divided into N+1 annular strips by N lines equally spaced $\Delta\eta = \frac{1}{N+1}$ apart. The value of η along any line is then $\eta_i = \frac{i-1}{N+1}$ $\quad i = 2, 3, \ldots, N, N+1 \qquad (53)$

i=1 corresponds to η=0, i=N+2 corresponds to η=1. Along any of the N lines the first order partial derivative with respect to η of a function $F(\eta,\theta)$ is approximated by the second order central difference formula

$$\frac{\partial F}{\partial \eta}(\eta,\theta) = \frac{F(\eta+\Delta\eta,\theta) - F(\eta-\Delta\eta,\theta)}{2\Delta\eta} \qquad (54)$$

Along the unit circle $\eta=1$. Here, since only values of η less than 1 are available, the partial with respect to η is approximated by the second order backward difference formula

$$\frac{\partial F(1,\theta)}{\partial \eta} = \frac{3F(1,\theta) - 4F(1-\Delta\eta,\theta) + F(1-2\Delta\eta,\theta)}{2\Delta\eta} \tag{55}$$

Using subscripts corresponding to the value of η_i desired, the finite difference formulae may be written

$$\left.\frac{\partial F}{\partial \eta}\right|_{\eta_i} = \frac{N+1}{2}\left(F_{i+i} - F_{i-i}\right) \tag{56}$$

for the central difference approximation and

$$\left.\frac{\partial F}{\partial \eta}\right|_{\eta=1} = \frac{N+1}{2}\left(3F_{N+2} - 4F_{N+1} + F_N\right) \tag{57}$$

for the backward difference approximation.

Finally, applying equations (56) and (57) to the governing equations (51) and (52) yields

$$\frac{\partial v_i}{\partial \theta} = \left[\left(\frac{Av_i^2+Du_i^2-B}{Au_i^2+Dv_i^2-B}\right) \frac{i-1}{2} \left(u_{i+i} - u_{i-i}\right)\right] + \left[\left(\frac{2Eu_i v_i}{Au_i^2+Dv_i^2-B}\right) \frac{i-1}{2} \left(v_{i+1} - v_{i-1}\right)\right] - u_i \tag{58}$$

for the equation of motion and

$$\frac{\partial u_i}{\partial \theta} = v_i - \frac{(i-1)}{2}\left(v_{i+1} - v_{i-1}\right) \tag{59}$$

for the irrotationality condition.

Vita

Michael Patrick Burke was born on 6 July 1954 in Milwaukee, Wisconsin. He attended Northwestern University and then the University of Wisconsin-Madison from which he received the degree of Bachelor of Science in Mechanical Engineering in May 1978. Upon graduation he received a reserve commission in the United States Air Force and was assigned to the Air Force Institute of Technology.

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